MIXED RATIO LIMIT THEOREMS FOR MARKOV PROCESSES

BY

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ABSTRACT

Let P be a conservative and ergodic Markov operator on $L_1(X,\Sigma,m)$. We give a sufficient condition for the existence of a decomposition $A_j \uparrow X$ such that for $0 \leqq f, g \in L_\infty(A_j)$ and any two probability measures μ and ν weaker than m

$$\sum_{n=1}^{N} \langle \nu P^n, g\rangle \Big/ \sum_{n=1}^{N} \langle \mu P^n, f\rangle \text{ converges to } \langle \lambda, g\rangle / \langle \lambda, f\rangle,$$

where λ is the σ-finite invariant measure (which necessarily exists). Processes recurrent in the sense of Harris are shown to have this decomposition, and an analytic proof of the convergence of

$$\sum_{n=1}^{N} P^n 1_A(x) \Big/ \sum_{n=1}^{N} P^n 1_B(y) \text{ to } \lambda(A)/\lambda(B)$$

is deduced for such processes.

1. Definitions and notations

Let (X, Σ, m) be a measure space with $m(X) = 1$. A *Markov process* is a positive contraction P on $L_1(X, \Sigma, m)$. P will be written to the right of its variable, while its adjoint, acting on $L_\infty(X, \Sigma, m)$, will be denoted by P and written to the left of its variable. Thus $\langle uP, f\rangle = \langle u, Pf\rangle$ for $f \in L_\infty$, $u \in L_1$.

By the Radon-Nikodym theorem P also acts on the Banach space of finite signed measures absolutely continuous with respect to $m: \mu P(A) = \int P 1_A d\mu$ for $\mu \ll m, A \in \Sigma$. The same formula defines μP for a σ-finite positive measure $\mu \ll m$. A positive σ-finite measure μ is *invariant* if $\mu P = \mu$.

The process is *conservative* if $m(A) > 0$ implies $\sum_{n=0}^{\infty} P^n 1_A(x) = \infty$ a.e. on A.

* This paper is a part of the author's Ph.D. thesis prepared at the Hebrew University of Jerusalem under the direction of Professor S. R. Foguel, to whom the author is grateful for his helpful advice and kind encouragement.

Received April 14, 1970.

The process is *conservative and ergodic* if $m(A) > 0$ implies $\sum_{n=0}^{\infty} P^n 1_A(x) = \infty$ a.e.

If $A \in \Sigma$, we define the operator T_A by $uT_A(x) = u(x)1_A(x)$, so $T_A f(x) = 1_A(x)f(x)$ and $\mu T_A(B) = \mu(A \cap B)$.

The complement of a set A is denoted by A'.

2. Ratio limit theorems

THEOREM 2.1. *Let P be a conservative and ergodic Markov process, and let $A \in \Sigma$ be with $m(A) > 0$. A necessary and sufficient condition for the convergence $\lim_{N\to\infty} \sum_{n=1}^{N} \nu P^n(A)/\sum_{n=1}^{N} \eta P^n(A) = 1$ for any two probability measures ν and $\eta \ll m$ is: There exists a probability measure $\mu \ll m$ such that*

$$\limsup_{N\to\infty} \left\| \sum_{n=1}^{N} P^n 1_A(x) \Big/ \sum_{n=1}^{N} \mu P^n(A) \right\|_{\infty} < \infty .$$

PROOF: Necessity: Take any fixed probability measure μ. $\sum_{n=1}^{\infty} \mu P^n(A) = \infty$ by ergodicity, so for $N \geqq N_0$ $\sum_{n=1}^{N} \mu P^n(A) > 0$. Define, for $N \geqq N_0$, $f_N(x) = \sum_{n=1}^{N} P^n 1_A(x)/ \sum_{n=1}^{N} \mu P^n(A)$. The sequence $\{f_N : N \geqq N_0\}$ defines a sequence of linear functionals on $L_1(X,\Sigma,m)$, since $f_N \in L_\infty(X,\Sigma,m)$. Every signed measure $\ll m$ is the difference of two positive measures $\ll m$, so by the given convergence, f_N is weak-* convergent to 1, hence $\{\| f_N \|_\infty : N \geqq N_0\}$ is bounded.

Sufficiency: For every $\nu \ll m$, the condition implies $\{ \sum_{n=1}^{N} \nu P^n(A)/ \sum_{n=1}^{N} \mu P^n(A) : N \geqq N_0\}$ is bounded.

Take a fixed probability measure ν_0, and let $\{N_j\}$ be any sequence such that $\sum_{n=1}^{N_j} \nu_0 P^n(A)/ \sum_{n=1}^{N_j} \mu P^n(A)$ converges. We have to show that the limit is one.

We define on $L_1(m)$, identified with the space of finite signed measures $\ll m$, a positive linear functional L by a Banach limit

$$L(\nu) = \mathrm{LIM} \left\{ \sum_{n=1}^{N_j} \nu P^n(A) \Big/ \sum_{n=1}^{N_j} \mu P^n(A) \right\}.$$

$$L(\nu P) = \mathrm{LIM} \left\{ \sum_{n=1}^{N_j} \nu P^{n+1}(A) \Big/ \sum_{n=1}^{N_j} \mu P^n(A) \right\}$$

$$= \mathrm{LIM} \left\{ \sum_{n=1}^{N_j} \nu P^n(A) \Big/ \sum_{n=1}^{N_j} \mu P^n(A) \right\} + \mathrm{LIM} \left\{ [\nu P^{N_j+1}(A) - \nu P(A)] \Big/ \sum_{n=1}^{N_j} \mu P^n(A) \right\}$$

$$= L(\nu).$$

The last term is zero since $\sum_{n=1}^{\infty} \mu P^n(A) = \infty$ and Banach limits preserve limits.

But $L(v) = \int g dv$ for some $g \in L_\infty(X, \Sigma, m)$, and $L(vP) = L(v)$ implies $Pg = g$. The process is conservative and ergodic, hence g is a constant [1, theorem III.A]. $g = \int g d\mu = L(\mu) = 1$, hence $L(v) = \int dv = v(X)$. Thus

$$1 = v_0(X) = L(v_0) = \lim_{N_j \to \infty} \sum_{n=1}^{N_j} v_0 P^n(A) \Big/ \sum_{n=1}^{N_j} \mu P^n(A).$$

Since this holds for every convergent subsequence, the sequence itself must converge to one.

REMARK. The conditions on A in the above theorem *do not* imply the existence of a σ-finite invariant measure: take a process with no such measure and $A = X$.

LEMMA 2.1. *If $B \in \Sigma$, then for $n \geqq 1$*

$$P^n = \sum_{j=1}^{n-1} (PT_{B'})^{j-1} PT_B P^{n-j} + (PT_{B'})^{n-1} P.$$

PROOF: An easy induction. For $n = 1$ the sum is zero, and $P = (PT_{B'})^0 P$. Using the induction hypothesis and $PT_B + PT_{B'} = P$ we get

$$P^{n+1} = P^n P = \sum_{j=1}^{n-1} (PT_{B'})^{j-1} PT_B P^{n+1-j} + (PT_{B'})^{n-1}(PT_B + PT_{B'})P$$

$$= \sum_{j=1}^{n} (PT_{B'})^{j-1} PT_B P^{n+1-j} + (PT_{B'})^n P.$$

LEMMA 2.2. *If $B \in \Sigma$, then for $n \geqq 1$*

$$\sum_{m=0}^{\infty} (PT_{B'})^m P^n 1_B \leqq n.$$

PROOF: For $n = 1$ we have

$$\sum_{m=0}^{N} (PT_{B'})^m P 1_B = \sum_{m=0}^{N} (PT_{B'})^m P(1 - T_{B'} 1) \leqq 1 - (PT_{B'})^{N+1} 1 \leqq 1$$

and letting $N \to \infty$ the result follows. Continuing by induction

$$\sum_{m=0}^{N} (PT_{B'})^m P^{n+1} 1_B = \sum_{m=0}^{N} (PT_{B'})^m PT_{B'} P^n 1_B +$$

$$+ \sum_{m=0}^{N} (PT_{B'})^m PT_B P^n 1_B \leqq n + 1.$$

The first term is less than n by the induction hypothesis, the second less than 1 since $T_B P^n 1_B \leqq 1_B$ and by our proof for $n = 1$. Letting $N \to \infty$ the result follows.

LEMMA 2.3. *If $A, B \in \Sigma$, $m(A) > 0$, $m(B) > 0$ and for some $K > 0$ and $\alpha > 0$ $\sum_{n=0}^{K-1} P^n 1_B \geqq \alpha 1_A$ then $\sum_{m=0}^{\infty} (PT_{B'})^m P 1_A \leqq K(K+1)/2\alpha$.*

PROOF:

$$\sum_{m=0}^{N} (PT_{B'})^m P 1_A \leqq \alpha^{-1} \sum_{m=0}^{N} (PT_{B'})^m P \sum_{n=0}^{K-1} P^n 1_B$$

$$= \alpha^{-1} \sum_{n=1}^{K} \sum_{m=0}^{N} (PT_{B'})^m P^n 1_B \leqq \alpha^{-1} \sum_{n=1}^{K} n = K(K+1)/2\alpha.$$

The last inequality is by Lemma 2.2. Let $N \to \infty$ to deduce the Lemma.

THEOREM 2.2. *Let P be an ergodic and conservative Markov process and $A \in \Sigma$ with $m(A) > 0$. If there exist a probability measure $\mu \ll m$, an integer $K > 0$, an $\alpha > 0$ and $0 < \varepsilon < 1$ such that*

$$\mu(B) \geqq \varepsilon \Rightarrow \sum_{n=0}^{K-1} P^n 1_B \geqq \alpha 1_A$$

then for every two probability measures ν and η

$$\lim_{N \to \infty} \sum_{n=1}^{N} \nu P^n(A) / \sum_{n=1}^{N} \eta P^n(A) = 1.$$

PROOF: We have to show that μ and A satisfy the condition of Theorem 2.1. Define $\delta = \varepsilon/(1-\varepsilon)$, hence $\delta > 0$. We denote $f_N = \sum_{n=1}^{N} P^n 1_A$, and $S_N = \sum_{n=1}^{N} \mu P^n(A) = \int f_N d\mu$.

The process is conservative and ergodic, hence $S_N \uparrow \infty$. We deal with $N \geqq N_0$ such that $S_N > 0$. Define $B_N = \{x : f_N(x) \leqq (1+\delta)S_N\}$.

$$S_N = \int f_N d\mu \geqq \int_{X-B_N} f_N d\mu \geqq (1+\delta)S_N \mu(X - B_N)$$

$$= (1+\delta)S_N[1 - \mu(B_N)].$$

Dividing by $S_N > 0$ and solving yields

$$\mu(B_N) \geqq 1 - 1/(1+\delta) = \delta/(1+\delta) = \varepsilon.$$

For short we shall denote B_N by B. Hence $\sum_{n=0}^{K-1} P^n 1_B \geqq \alpha 1_A$ by our assumptions.

Using Lemma 2.1 we have

$$\sum_{n=1}^{N} P^n 1_A = \sum_{n=1}^{N} \sum_{j=1}^{n-1} (PT_{B'})^{j-1} PT_B P^{n-j} 1_A + \sum_{n=1}^{N} (PT_{B'})^{n-1} P1_A$$

$$= \sum_{j=1}^{N-1} \sum_{n=j+1}^{N} (PT_{B'})^{j-1} PT_B P^{n-j} 1_A + \sum_{n=1}^{N} (PT_{B'})^{n-1} P1_A$$

$$\leqq \sum_{j=1}^{N-1} (PT_{B'})^{j-1} PT_B \sum_{n=1}^{N} P^n 1_A + \sum_{n=1}^{N} (PT_{B'})^{n-1} P1_A .$$

But $T_B \sum_{n=1}^{N} P^n 1_A = T_B f_N \leqq (1+\delta) S_N 1_B$. Hence

$$\sum_{n=1}^{N} P^n 1_A \leqq (1+\delta) S_N \sum_{j=1}^{N-1} (PT_{B'})^{j-1} P1_B + \sum_{n=1}^{N} (PT_{B'})^{n-1} P1_A .$$

$\sum_{j=1}^{N-1}(PT_{B'})^{j-1}P1_B \leqq 1$ by Lemma 2.2 with $n = 1$. $\sum_{n=1}^{N}(PT_{B'})^{n-1}P1_A \leqq K(K+1)/2\alpha$ by Lemma 2.3. (K and α do not depend on $N \geqq N_0$; only B does). Hence

$$\sum_{n=1}^{N} P^n 1_A \leqq (1+\delta) S_N + K(K+1)/2\alpha .$$

$S_N \geqq S_{N_0}$, so by dividing by S_N we have

$$\Big\| \sum_{n=1}^{N} P^n 1_A \Big/ \sum_{n=1}^{N} \mu P^n(A) \Big\|_\infty \leqq K(K+1)/(2\alpha S_{N_0}) + 1 + \delta$$

hence the conditions of Theorem 2.1 are satisfied. Q.E.D.

REMARK. The definition of B_N in the proof follows [9, p. 49], where stronger assumptions are used, with probabilistic arguments replacing our analytic Theorem 2.1. The theorem there is proved only for Harris processes, which we discuss in the next section.

THEOREM 2.3. *Under the conditions of Theorem 2.2, there exists a σ-finite invariant measure λ equivalent to m, with $\lambda(A) < \infty$, and there exists a sequence of sets $A_j \uparrow X$ with $A_0 = A$, $\lambda(A_j) < \infty$, such that for and two probability measures η and $\nu \ll m$ and $0 \leqq f, g \in L_\infty(A_j)$*

$$\sum_{n=1}^{N} \langle \nu P^n, f \rangle \Big/ \sum_{n=1}^{N} \langle \eta P^n, g \rangle \to \langle \lambda, f \rangle / \langle \lambda, g \rangle .$$

PROOF: Let $B \in \Sigma$ with $\mu(B) \geqq \varepsilon$.

$$\alpha \sum_{n=1}^{N} P^n 1_A \leqq \sum_{n=1}^{N} \sum_{j=0}^{K-1} P^{n+j} 1_B \leqq K \sum_{n=1}^{N+K} P^n 1_B .$$

Hence $\liminf_{N\to\infty} \sum_{n=1}^{N} \mu P^n(B) / \sum_{n=1}^{N} \mu P^n(A) \geqq \alpha/K > 0$ ($\sum_{n=1}^{\infty} \mu P^n(A) = \infty$ so $\sum_{n=1}^{N+K} \mu P^n(A) / \sum_{n=1}^{N} \mu P^n(A) \to 1$).

By the corollary to theorem 3 of Horovitz [3] the existence of λ follows, and $0 < \lambda(A) < \infty$.

The set A satisfies condition I of [2], i.e. if $m(E) > 0$ there exist $M = M(E)$ and $\beta = \beta(E) > 0$ such that $\sum_{n=0}^{M} P^n 1_E \geqq \beta 1_A$. The proof is given in Remark (2) on Condition I in [2].

Define $A_j = \{x: \sum_{n=0}^{j} P^n 1_A(x) \geqq 1\}$. By ergodicity and conservativity $A_j \uparrow X$. Since $\sum_{n=0}^{j} P^n 1_A \geqq 1_{A_j}$ we have $\lambda(A_j) \leqq (j+1)\lambda(A) < \infty$.

Each set A_j satisfies condition I of [2] (defined above) by Remark (3) on this condition in [2]. Theorem 1 of [2] now says that for $0 \leqq f,\ g \in L_\infty(A_j)$

$$\lim_{N\to\infty} \sum_{n=1}^{N} \langle \nu P^n, f\rangle / \sum_{n=1}^{N} \nu P^n(A_j) = \langle \lambda, f\rangle / \lambda(A_j)$$

$$\lim_{N\to\infty} \sum_{n=1}^{N} \langle \eta P^n, g\rangle / \sum_{n=1}^{N} \eta P^n(A_j) = \langle \lambda, g\rangle / \lambda(A_j).$$

Since also $1_A \in L_\infty(A_j)$ we have

$$\lim_{N\to\infty} \sum_{n=1}^{N} \langle \nu P^n, f\rangle / \sum_{n=1}^{N} \nu P^n(A) = \langle \lambda, f\rangle / \lambda(A)$$

$$\lim_{N\to\infty} \sum_{n=1}^{N} \eta P^n(A) \,/\, \sum_{n=1}^{N} \langle \eta P^n, g\rangle = \lambda(A) / \langle \lambda, g\rangle$$

Using Theorem 2.2 we have $\sum_{n=1}^{N} \nu P^n(A) / \sum_{n=1}^{N} \eta P^n(A) \to 1$. Hence

$$\frac{\sum_{n=1}^{N} \langle \nu P^n, f\rangle}{\sum_{n=1}^{N} \langle \eta P^n, g\rangle} = \frac{\langle \sum_{n=1}^{N} \nu P^n, f\rangle}{\sum_{n=1}^{N} \nu P^n(A)} \; \frac{\sum_{n=1}^{N} \nu P^n(A)}{\sum_{n=1}^{N} \eta P^n(A)} \; \frac{\sum_{n=1}^{N} \eta P^n(A)}{\sum_{n=1}^{N} \langle \eta P^n, g\rangle} \to \frac{\langle \lambda, f\rangle \lambda(A)}{\lambda(A) \langle \lambda, g\rangle}$$

and the theorem is proved.

Remarks. (1) If $\sum_{n=1}^{N} \mu P^n(B) / \sum_{n=1}^{N} \mu P^n(A)$ converges for every $B \subseteq A$, then there exists a σ-finite invariant measure λ, by Lemma 2.2 of [6], and $\lambda(A) < \infty$.

(2) If there exists a σ-finite invariant measure λ with $0 < \lambda(A) < \infty$, the conditions of Theorem 2.2 are *not necessarily* satisfied. Krengel [5, example 1.1] has an example of a conservative and ergodic Markov chain on a countable space where a set A with $\lambda(A) < \infty$ has two probability measures μ, ν with $d\mu/d\lambda, d\nu/d\lambda \in L_\infty(B)$, $\lambda(B) < \infty$, and

$$\limsup_{N\to\infty} \sum_{n=1}^{N} \mu P^n(A)/\sum_{n=1}^{N} \nu P^n(A) = \infty; \ \liminf_{N\to\infty} \sum_{n=1}^{N} \mu P^n(A)/\sum_{n=1}^{N} \nu P^n(A) = 0.$$

(3) If $\lambda(X) < \infty$ the conclusions of Theorem 2.2 follow immediately from the ergodic theorem, for *every* $A \in \Sigma$ with $m(A) > 0$.

3. Applications to Harris processes

DEFINITION 3.1. An ergodic and conservative process is called a *Harris process* if for some $n > 0$ there exists a $\Sigma \times \Sigma$ measurable function $0 \neq q(x,y) \geqq 0$ with $\int q(x,y)m(dy) \leqq 1$ a.e. and $0 \leqq Q \leqq P^n$, where Q is the integral operator $uQ(y) = \int u(x)q(x,y)m(dx)$. (See [1, theorem V.F]).

By [1, theorem V.C] there is a decomposition $P^n = Q_n + R_n$, where Q_n is an integral operator and if K is an integral operator with $0 \leqq K \leqq R_n$, then $K = 0$. A Harris process has $Q_n \neq 0$ for some $n > 0$, by definition.

LEMMA 3.1. *Let P be an ergodic Harris process. Then there exists a set $A \in \Sigma$ satisfying the hypothesis of Theorem 2.2. Hence the conclusions of Theorem 2.3 hold.*

For the sake of completeness we repeat the proof of [3].

Suppose $Q_k \neq 0$, and let its kernel be $q_k(x,y)$. Let $E_x = \{y : q_k(x,y) \geqq \beta\}$. For some $\beta > 0$, we have

$$0 < m^2\{(x,y) : q_k(x,y) \geqq \beta\} = \int m(E_x)m(dx).$$

Thus for some $\delta > 0$ $A = \{x : m(E_x) \geqq \delta\}$ has $m(A) > 0$. Define $\varepsilon = 1 - \delta/2$. $m(B) \geqq \varepsilon$ implies $m(B \cap E_x) > \delta/2$ for $x \in A$. Hence for $x \in A$

$$P^k 1_B(x) \geqq Q_k 1_B(x) = \int_B q_k(x,y)m(dy) \geqq \int_{B \cap E_x} q_k(x,y)m(dy) \geqq \beta\delta/2.$$

Hence $P^k 1_B \geqq \beta\delta/2\, 1_A$, so A satisfies our condition with $\alpha = \beta\delta/2$ and $K = k + 1$.

Q.E.D.

REMARK. Other sets having the desired property exist by [7, theorem 4] and [8, theorem 2.1], but those proofs are much more difficult. Metivier's results [7, theorem 6] impose some conditions on the measures μ and ν, removed by Theorem 2.3.

As P is not assumed to be induced by a transition probability, μP is defined only for $\mu \ll m$. Thus we cannot put Dirac measures instead of μ. The fact that the results are still true is now shown.

THEOREM 3.1. *Let P be an ergodic Harris process, let λ be its σ-finite invariant measure. If $E, F \in \Sigma$ with $0 < \lambda(F) < \infty$, then there exists a λ-null set Z such that for $x, y \notin Z$*

$$\lim_{N\to\infty} \sum_{n=1}^{N} P^n 1_E(x) / \sum_{n=1}^{N} P^n 1_F(y) = \lambda(E)/\lambda(F).$$

PROOF: Let $A \in \Sigma$ be a set satisfying the hypothesis of Theorem 2.2 (it exists by the previous lemma). Let $S_N = \sum_{n=1}^{N} mP^n(A)$. For $N \geqq N_0$ $S_N > 0$, and the proof of Theorem 2.2 shows that $\sum_{n=1}^{N} P^n 1_A(x)/S_N \leqq M$ a.e., independently of N.

If $P^k = Q_k + R_k$, then by theorem V.E of [1] $R_k 1(x) \downarrow 0$ a.e., so for $k \geqq k_0$ $Q_k \neq 0$, with kernel $q_k(x,y)$.

The functions appearing in the sequel are defined only a.e., and satisfy some relations a.e. Since there are only countably many functions and countably many relations, we may take a fixed version of each function and exclude a set Z_0 with $\lambda(Z_0) = 0$ such that if $x \notin Z_0$ all the functions are defined and satisfy all the needed relations (equalities, inequalities and convergence).

Take $x \notin Z_0$. Given $\delta > 0$, we can choose k so large such that $R_k 1(x) \leqq \delta$, and hence $|Q_k 1(x) - 1| \leqq \delta$.

Let such k be fixed, and define

$$v_x(A) = \int_A q_k(x,y) m(dy), \text{ so } v_x \ll m, \text{ and } v_x(X) = Q_k 1(x).$$

$$v_x P^n(A) = \int P^n 1_A(y) dv_x(y) = \int q_k(x,y) P^n 1_A(y) m(dy) = Q_k P^n 1_A(x).$$

By Theorem 2.2 applied to v_x and by the last equality

$$Q_k \sum_{n=1}^{N} P^n 1_A(x)/S_N = \sum_{n=1}^{N} v_x P^n(A)/S_N \underset{N\to\infty}{\to} v_x(X) = Q_k 1(x).$$

$$\sum_{n=k+1}^{N+k} P^n 1_A(x) = P^k \sum_{n=1}^{N} P^n 1_A(x) = (Q_k + R_k) \sum_{n=1}^{N} P^n 1_A(x)$$

and hence

$$\left| \sum_{n=1}^{N} P^n 1_A(x)/S_N - 1 \right| \leqq \left| \left\{ \sum_{n=1}^{N} P^n 1_A(x) - \sum_{n=k+1}^{N+k} P^n 1_A(x) \right\} / S_N \right|$$

$$+ \left| Q_k \sum_{n=1}^{N} P^n 1_A(x)/S_N - Q_k 1(x) \right| + \left| Q_k 1(x) - 1 \right| + \left| R_k \sum_{n=1}^{N} P^n 1_A(x)/S_N \right|.$$

The second term tends to zero as $N \to \infty$. The sum of the two last terms is bounded by $\delta + M\delta$ and the first term by $2k/S_N$. Thus

$$\limsup_{N\to\infty} \left| \sum_{n=1}^{N} P^n 1_A(x)/S_N - 1 \right| \leqq \delta + M\delta.$$

This being done for any $\delta > 0$ yields

$$\sum_{n=1}^{N} P^n 1_A(x)/S_N \to 1, \text{ and hence for } x, y \notin Z_0$$

$$\sum_{n=1}^{N} P^n 1_A(x) \Big/ \sum_{n=1}^{N} P^n 1_A(y) \to 1.$$

By the Chacon-Ornstein theorem [1, III.E] (applied to the adjoint process P^* on $L_1(X,\Sigma,\lambda)$), if $\lambda(E) < \infty$, then for $x, y \notin Z$ (and $\lambda(Z) = 0$)

$$\frac{\sum_{n=1}^{N} P^n 1_E(x)}{\sum_{n=1}^{N} P^n 1_F(y)} = \frac{\sum_{n=1}^{N} P^n 1_E(x)}{\sum_{n=1}^{N} P^n 1_A(x)} \frac{\sum_{n=1}^{N} P^n 1_A(x)}{\sum_{n=1}^{N} P^n 1_A(y)} \frac{\sum_{n=1}^{N} P^n 1_A(y)}{\sum_{n=1}^{N} P^n 1_F(y)} \to \frac{\lambda(E)}{\lambda(F)}.$$

If $\lambda(E) = \infty$ we take $E_j \uparrow E$ with $\lambda(E_j) < \infty$ and use the above result. Q.E.D.

REMARK. This theorem is due to Jain [4], but his proof is based on probabilistic results. Other proofs are due to Metivier [7] and Levitan [9].

If P is an ergodic Harris process, so is its adjoint process P^*. The roles of measures and functions are interchanged when conditions (and results) for P^* are expressed in terms of P. In the next theorem we interpret Theorem 2.3 for P^* in terms of P. We see that the conditions on the sets can be relaxed if some restrictions are imposed on the measures.

THEOREM 3.2. *Let P be an ergodic Harris process and λ its σ-finite invariant measure. There exists a sequence $A_j \uparrow X$ with $\lambda(A_j) < \infty$ such that for any two sets E and F in Σ with $\lambda(E), \lambda(F) < \infty$ and any two probability measures $\mu, \nu \ll m$ with $d\mu/d\lambda, d\nu/d\lambda \in L_\infty(A_j)$*

$$\sum_{n=1}^{N} \mu P^n(E) / \sum_{n=1}^{N} \nu P^n(F) \to \lambda(E)/\lambda(F).$$

PROOF. The adjoint process P^* is also an ergodic Harris process [1, chapter VII]. Let A be the set for P^* guaranteed by Lemma 3.1 and A_j the sequence of Theorem 2.3 (for P^*). Let $d\mu/d\lambda = f$, $d\nu/d\lambda = g$. If $d\mu_0/d\lambda = 1_E$, $d\nu_0/d\lambda = 1_F$, then by Theorem 2.3

$$\sum_{n=1}^{N} \mu P^n(E) \,/\, \sum_{n=1}^{N} \nu P^n(F) = \sum_{n=1}^{N} \langle \mu_0 P^{*n}, f \rangle \,/\, \sum_{n=1}^{N} \langle \nu_0 P^{*n}, g \rangle$$

$$\underset{N \to \infty}{\to} \mu_0(X) \langle \lambda, f \rangle / \nu_0(X) \langle \lambda, g \rangle = \lambda(E)/\lambda(F).$$

REMARKS. (1) Krengel's example cited at the end of §2 shows that there may exist a probability measure η and sets E, B such that the ratios $\sum_{n=1}^{N} \eta P^n(E) / \sum_{n=1}^{N} \eta P^n(B)$ do not converge: Take for P the adjoint of the above example, $d\eta/d\lambda = 1_A$ where A is defined in that example. If $d\mu/d\lambda = f$ and $d\nu/d\lambda = g$ for that example,

$$\sum_{n=1}^{N} \langle \eta P^n, f \rangle \,/\, \sum_{n=1}^{N} \langle \eta P^n, g \rangle = \sum_{n=1}^{N} \mu P^{*n}(A) / \sum_{n=1}^{N} \nu P^{*n}(A)$$

does not converge, and hence there is an $E \subset B$ ($f, g \in L_\infty(B)$ by the construction, and $\lambda(B) < \infty$) for which $\sum_{n=1}^{N} \eta P^n(E) / \sum_{n=1}^{N} \eta P^n(B)$ does not converge.

(2) In Theorem 3.2 we have removed the "boundedness" assumption of Metivier on the sets E and F [7]. In Theorem 2.3 we have such an assumption on the sets, but no restrictions on the measures, while Metivier restricts both the sets *and* the measures. Furthermore, in Theorem 2.3 we do not assume that P is a Harris process.

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